On the irrationality measure of certain numbers

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Abstract

The paper presents upper estimates for the irrationality measure and the non-quadraticity measure for the numbers $\alpha_k = \sqrt{2k+1}\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}$, $k \in \mathbb{N}$.

For any irrational α, *the irrationality measure* can be defined as the exact upper bound on the numbers κ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < q^{-\kappa}$$

has infinitely many rational solutions $\frac{p}{q}$. By $\mu(\alpha)$ denote the irrationality measure of α.

The non-quadraticity measure can be defined for any real α which isn't a root of a quadratic equation as the exact upper bound on the numbers κ such that the inequality

$$|\alpha - \beta| < H^{-\kappa}(\beta)$$

has infinitely many rational solutions in quadratic irrationalities β. Here $H(\beta)$ is the height of the characteristic polynomial of β (taking an irreducible integer polynomial with one of the roots equal to β, H the largest absolute value of this polynomial's coefficients). By $\mu_2(\alpha)$ denote the non-quadraticity measure of α.

We are going to preset improved bounds on the irrationality measure and the non-quadraticity measure for the numbers

$$\alpha_k = \sqrt{2k+1}\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}, \text{ where } k \in \mathbb{N}. \tag{1}$$

This paper is, in a sense, a continuation of the paper [2] by M. Bashmakova: the same integral is considered, but the denominator is estimated more accurately by using a certain coefficient symmetry. Some earlier estimates for the irrationality measure of the numbers investigated by the author have been obtained by A. Heimonen, T. Matala-Aho, and K. Väänänen [6], M. Hata [4], G. Rhin [10], E. Salnikova [11], M. Bashmakova [1], [2].

Some of the numerical results obtained in this paper have been summarized in the table below:

k	$\mu(\alpha_k) \leqslant$	$\mu_2(\alpha_k) \leqslant$	k	$\mu(\alpha_k) \leqslant$	$\mu_2(\alpha_k) \leqslant$	k	$\mu(\alpha_k) \leqslant$	$\mu_2(\alpha_k) \leqslant$
3	6.64610...	—	7	5.45248...	—	10	3.45356...	10.0339...
5	5.82337...	—	8	3.47834...	10.9056...	11	5.08120...	—
6	3.51433...	12.4084...	9	5.23162...	—	12	3.43506...	9.46081...

Let n be an odd positive integer, and let a, b be fixed positive integers (where b is odd) such that $b > 4a$. Define a polynomial as follows:

$$A(x) = \binom{x+(b-2a)n}{(b-4a)n}\binom{x+(b-a)n}{(b-2a)n}\binom{x+bn}{bn} =$$

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$$= \frac{(x+2an+1)\dots(x+(b-2a)n)}{((b-4a)n)!} \cdot \frac{(x+an+1)\dots(x+(b-a)n)}{((b-2a)n)!} \cdot \frac{(x+1)\dots(x+bn)}{(bn)!}.$$

Consider an integral

$$I(z) = \frac{z^{-\frac{bn+1}{2}}}{2\pi i} \int_L A(\zeta) \left(\frac{\pi}{\sin \pi\zeta}\right)^3 (-z)^{-\zeta} d\zeta,$$

where $z \neq 0$, the vertical line L is given by the equation $\mathfrak{Re}\,\zeta = C$, where $-(b-2a)n < C < -2an-1$, and this line is traversed from bottom to top. We also suppose that $(-z)^{-\zeta} = e^{-\zeta \ln(-z)}$, where the branch of the logarithm $\ln(-z) = \ln|z| + i \arg z + i\pi$ is chosen so that $|\arg z| < \pi$.

Statement 1. *For all $z \in \mathbb{C}$ such that $0 < |z| < 1$ we have*

$$I(z) = -\frac{1}{2}U(z)\ln^2 z + V(z)\ln z - \frac{1}{2}W(z) - i\pi(U(z)\ln z - V(z)), \tag{2}$$

where the functions $U(z), V(z), W(z) \in \mathbb{Q}(z)$ are defined for $|z| < 1$ by the following equations:

$$\begin{aligned} U(z) &= -z^{-\frac{bn+1}{2}} \sum_{k=bn+1}^{\infty} A(-k)z^k, \\ V(z) &= -z^{-\frac{bn+1}{2}} \sum_{k=(b-a)n+1}^{\infty} A'(-k)z^k, \\ W(z) &= -z^{-\frac{bn+1}{2}} \sum_{k=(b-2a)n+1}^{\infty} A''(-k)z^k. \end{aligned} \tag{3}$$

The proof of this statement (in a somewhat different form) has been given by Yuri Nesterenko [7]. He has also proved the following lemma (see Lemma 1 in [7]).

Lemma 1. *Let $P(x) \in \mathbb{C}[x]$ be a polynomial of degree d. Then for all z, $|z| < 1$, we have*

$$-\sum_{k=1}^{\infty} P(-k)z^k = \sum_{j=0}^{d} c_j \left(\frac{z}{z-1}\right)^{j+1},$$

where

$$c_j = \sum_{k=1}^{k=j+1} (-1)^{k-1} P(-k) \binom{j}{k-1}.$$

It has also been shown (see Statement 2 and Lemma 1 in [2]) that

$$\begin{gathered} U(z) = U\left(\tfrac{1}{z}\right) = \hat{U}\left(z + \tfrac{1}{z}\right), V(z) = -V\left(\tfrac{1}{z}\right) = \left(z - \tfrac{1}{z}\right)\hat{V}\left(z + \tfrac{1}{z}\right), \\ W(z) = W\left(\tfrac{1}{z}\right) = \hat{W}\left(z + \tfrac{1}{z}\right), \text{ where } \hat{U}(z), \hat{V}(z), \hat{W}(z) \in \mathbb{Q}(z). \end{gathered} \tag{4}$$

If the number x satisfies $x + \frac{1}{x} \in \mathbb{Q}$, then we obtain $U(x), \frac{V(x)}{x-1/x}, W(x) \in \mathbb{Q}$.

Now consider the values of the integral at the following points:

$$x_k = \frac{k+1-\sqrt{2k+1}}{k} = \frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}, \text{ where } k \in \mathbb{N}. \tag{5}$$

In this case we have $x_k + \frac{1}{x_k} = \frac{2k+2}{k}$ and $x_k - \frac{1}{x_k} = -2\frac{\sqrt{2k+1}}{k}$. This easily leads to

$$U(x_k), \sqrt{2k+1}V(x_k), W(x_k) \in \mathbb{Q}. \tag{6}$$

Let us define Ω as the set of $0 \leqslant y < 1$ such that for all $x \in \mathbb{R}$ the inequality

$$\begin{gathered}([x-2ay]-[x-(b-2a)y]-[(b-4a)y])+\\+([x-ay]-[x-(b-a)y]-[(b-2a)y])+([x]-[x-by]-[by]) \geqslant 1\end{gathered} \tag{7}$$

is satisfied.

The set Ω is a union of points, as well as closed, half-open and open intervals. Clearly, the points of the interval $\left[0; \frac{1}{b}\right)$ do not belong to Ω, and thus the set of primes $p > \sqrt{bn}$ such that $\left\{\frac{n}{p}\right\} \in \Omega$ is finite. Denote the product of these primes as Δ, and denote as Δ_1 the product of the primes $p > (b-2a)n$ satisfying $\left\{\frac{n}{p}\right\} \in \Omega$.

Let d_n be the least common multiple of $1, 2, \ldots, n$.

Let us introduce the following rational numbers:

$$R_{k,n} = \begin{cases} m^{-\frac{bn+1}{2}}, & \text{if } k = 2m; \\ 2^{\frac{3(b-2a)n+1}{2}} k^{-\frac{bn+1}{2}}, & \text{if } k = 2m-1. \end{cases}$$

$$S_{k,n} = \begin{cases} m^{-\frac{(b-2a)n+1}{2}}, & \text{if } k = 2m; \\ 2^{\frac{3(b-2a)n+1}{2}} k^{-\frac{(b-2a)n+1}{2}}, & \text{if } k = 2m-1. \end{cases}$$

$$T_{k,n} = \begin{cases} m^{-\frac{(b-4a)n+1}{2}}, & \text{if } k = 2m; \\ 2^{\frac{3(b-2a)n+1}{2}} k^{-\frac{(b-4a)n+1}{2}}, & \text{if } k = 2m-1. \end{cases}$$

Lemma 2. *Let x_k be defined by (5), then we have*

$$A = R_{k,n}U(x_k) \in \mathbb{Z},\ B = S_{k,n}\frac{d_{bn}}{\Delta}V(x_k)\sqrt{2k+1} \in \mathbb{Z},\ C = T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}W(x_k) \in \mathbb{Z}.$$

Proof. Let us prove the statement for B. The statements for A and C can be proved by the same argument.

It suffices to show that $B^2 \in \mathbb{K}$, where $\mathbb{K}$ is the ring of algebraic integers, which would imply that B is also an algebraic integer. Then from the property (6) we could say that $B \in \mathbb{Q}$, and consequently $B \in \mathbb{Z}$.

Let $A_1(z)$ be a polynomial of degree $3(b-2a)n$ such that $A_1(z) = A(z-an)$. Then it's easy to see that

$$A_1'(-1) = \cdots = A_1'(-(b-2a)n) = A'(-an-1) = \cdots = A'(-(b-a)n) = 0. \tag{8}$$

Let us rewrite (3). Defining l as $l = k - an$ and applying (8), we obtain

$$\begin{aligned}V(z) &= -z^{-\frac{bn+1}{2}} \sum_{k=(b-a)n+1}^{+\infty} A'(-k)z^k = -z^{-\frac{bn+1}{2}} \sum_{k=an+1}^{+\infty} A'(-k)z^k = \\ &= -z^{-\frac{bn+1}{2}+an} \sum_{k=an+1}^{+\infty} A'(-k)z^{k-an} = -z^{-\frac{bn+1}{2}+an} \sum_{l=1}^{+\infty} A'(-l-an)z^l = -z^{-\frac{bn+1}{2}+an} \sum_{l=1}^{+\infty} A_1'(-l)z^l.\end{aligned}$$

Then by Lemma 1 we have

$$V(z)\;=\;z^{-\frac{bn+1}{2}+an}\sum_{j=0}^{3(b-2a)n-1}b_j\left(\frac{z}{z-1}\right)^{j+1},\text{ where } b_j\;=\;\sum_{k=1}^{j+1}(-1)^{k-1}A_1'(-k)\binom{j}{k-1}.$$

Lemma 4 of [7] states that

$$\frac{d_{bn}}{\Delta}A_1'(-k)\in\mathbb{Z},\text{ where } 1\le k\le 3(b-2a)n,\text{ i.e. }\frac{d_{bn}}{\Delta}b_j\in\mathbb{Z}$$

From (8) it follows that

$$V(z)=z^{-\frac{bn+1}{2}+an}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_j\left(\frac{z}{z-1}\right)^{j+1}=$$

$$=z^{-\frac{bn+1}{2}+an}\left(\frac{z}{z-1}\right)^{(b-2a)n+1}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_j\left(\frac{z}{z-1}\right)^{j-(b-2a)n}.$$

It is true that $\frac{x_k}{x_k-1}=\frac{1-\sqrt{2k+1}}{2}=t_1$ and $\frac{1}{1-x_k}=\frac{1+\sqrt{2k+1}}{2}=t_2$ are the solutions of the equation $t^2-t-\frac{k}{2}=0$. Clearly, they must be algebraic numbers.

Applying (4) yields that

$$-\left(V(x_k)\right)^2=V(x_k)V\left(\frac{1}{x_k}\right)=(t_1t_2)^{(b-2a)n+1}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_jt_1^{j-(b-2a)n}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_jt_2^{j-(b-2a)n}=$$

$$=\left(\frac{k}{2}\right)^{(b-2a)n+1}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_jt_1^{j-(b-2a)n}\sum_{j=(b-2a)n}^{3(b-2a)n-1}b_jt_2^{j-(b-2a)n}. \tag{9}$$

If k is even, then for all positive integers N we have $t_i^N\in\mathbb{K}$ since $t_i\in\mathbb{K}$.

If k is odd, then for all positive integers N we can write $2^{\left[\frac{N+1}{2}\right]}t_i^N\in\mathbb{K}$ since $(t_i)^{2N}=\left(\frac{k+1\pm\sqrt{2k+1}}{2}\right)^N$ and $(t_i)^{2N+1}=\frac{1\pm\sqrt{2k+1}}{2}\left(\frac{k+1\pm\sqrt{2k+1}}{2}\right)^N$.

Let us consider the following two cases:

1. $k=2m$. Then by (9) we have

$$B^2=-\left(\sum_{j=(b-2a)n}^{3(b-2a)n-1}\left(\frac{d_{bn}}{\Delta}b_j\right)t_1^{j-(b-2a)n}\right)\left(\sum_{j=(b-2a)n}^{3(b-2a)n-1}\left(\frac{d_{bn}}{\Delta}b_j\right)t_2^{j-(b-2a)n}\right)(2k+1)\in\mathbb{K}.$$

2. $k=2m-1$. Then (9) yields that

$$B^2=-\left(\sum_{j=(b-2a)n}^{3(b-2a)n-1}\left(\frac{d_{bn}}{\Delta}b_j\right)2^{(b-2a)n}t_1^{j-(b-2a)n}\right)\times$$

$$\times\left(\sum_{j=(b-2a)n}^{3(b-2a)n-1}\left(\frac{d_{bn}}{\Delta}b_j\right)2^{(b-2a)n}t_2^{j-(b-2a)n}\right)(2k+1)\in\mathbb{K}.$$

This concludes the proof of the lemma. □

We proceed by formulating several known results, which have been stated in [2] as Lemmas 5 and 6.

Statement 2. *Let $x \in \mathbb{R}$, $0 < x < 1$. If the equation*

$$\frac{z(z-a)(z-2a)}{(z-(b-2a))(z-(b-a))(z-b)} = \frac{1}{x}$$

has a unique solution $z_0 > b$, then

$$\lim_{n\to+\infty} \frac{1}{n} \ln |U(x)| = M_1 = \ln \left(\frac{(z_0-(b-2a))^{b-2a}(z_0-(b-a))^{b-a}(z_0-b)^b}{(z_0-2a)^{2a}(z_0-a)^a(b-4a)^{b-4a}(b-2a)^{b-2a}b^b} \right) - \frac{b}{2} \ln x.$$

Statement 3. *Denote*

$$M_1 = \ln \frac{|z_1+(b-2a)|^{b-2a}|z_1+(b-a)|^{b-a}|z_1+b|^b}{|z_1+2a|^{2a}|z_1+a|^a(b-4a)^{b-4a}(b-2a)^{b-2a}b^b} - \frac{b}{2}\ln x,$$

where z_1 is the complex root of the equation

$$\frac{(z+2a)(z+a)z}{(z+(b-2a))(z+(b-a))(z+b)} = \frac{1}{x}$$

satisfying the condition $\mathfrak{Im}\, z_1 > 0$. Then we have

$$\limsup_{n\to+\infty} \frac{1}{n} \ln |I(x)| \leqslant M_2.$$

To compute

$$N_1 = \lim_{n\to\infty} \frac{1}{n} \ln \frac{d_{bn}}{\Delta} \qquad \text{and} \qquad N_2 = \lim_{n\to\infty} \frac{1}{n} \ln d_{(b-2a)n}\Delta_1 \frac{d_{bn}}{\Delta}, \tag{10}$$

we are going to use Lemma 6 from [7].

Lemma 3. *Let u, v be real numbers such that $0 < u < v < 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{u \le \left\{\frac{n}{p}\right\} < v} \ln p = \psi(v) - \psi(u),$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the logarithmic derivative of the gamma function, and the sum is taken over all primes p such that the fractional part $\left\{\frac{n}{p}\right\}$ lies in the given range.

Let us formulate a lemma by Hata (see [3], Lemma 2.1), which will allow us to prove the principal theorem of this paper.

Lemma 4. *Let $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, let α be an irrational number, and let $l_n = q_n\alpha + p_n$, where $q_n, p_n \in \mathbb{Z}$ and*

$$\lim_{n\to\infty} \frac{1}{n} \ln |q_n| = \sigma, \quad \limsup_{n\to\infty} \frac{1}{n} \ln |l_n| \le -\tau, \quad \sigma, \tau > 0.$$

Then

$$\mu(\alpha) \le 1 + \frac{\sigma}{\tau}.$$

Now we can state the principal theorem, which will be proved by applying the Hata's lemma to the sequence

$$E_n = S_{k,n}\frac{d_{bn}}{\Delta}\left(-\frac{\mathfrak{Im}(I(x_k))}{\pi}\sqrt{2k+1}\right) = S_{k,n}\frac{d_{bn}}{\Delta}U(x_k)\alpha_k - S_{k,n}\frac{d_{bn}}{\Delta}V(x_k)\sqrt{2k+1} = P_n\alpha_k + Q_n.$$

By Lemma 2, we have $P_n, Q_n \in \mathbb{Z}$. Clearly, we can also write

$$K_1 = \lim_{n\to\infty}\frac{1}{n}\ln(S_{k,n}) = \begin{cases} -\dfrac{b-2a}{2}\ln m, & \text{if } k = 2m; \\ -\dfrac{b-2a}{2}\ln k + \dfrac{3(b-2a)}{2}\ln 2, & \text{if } k = 2m-1. \end{cases}$$

Theorem 1. *Assume that $a, b \in \mathbb{N}$ satisfy $b > 4a$, x is defined by (5), the numbers M_1 and M_2 are defined by Statements 2 and 3, the set Ω is defined by (7), and N_1 is defined by (10).*

If $M_2 + K_1 + N_1 < 0$, we have

$$\mu(\alpha_k) \le 1 - \frac{\lim\limits_{n\to\infty} P_n}{\limsup\limits_{n\to\infty} E_n} \le 1 - \frac{M_1 + K_1 + N_1}{M_2 + K_1 + N_1}.$$

We are going to formulate another lemma by Hata (see [5], Lemma 2.3), which will allow us to prove another theorem.

Lemma 5. *Let $n \in \mathbb{N}$, and assume that $\alpha \in \mathbb{R}$ is not a quadratic irrationality (i.e., not a root of a quadratic integer polynomial). Take $l_n = q_n\alpha + p_n$, $m_n = q_n\alpha^2 + r_n$, where $q_n, p_n, r_n \in \mathbb{Z}$, and assume*

$$\lim_{n\to\infty}\frac{1}{n}\ln|q_n| = \sigma, \quad \max\left\{\limsup_{n\to\infty}\frac{1}{n}\ln|l_n|, \limsup_{n\to\infty}\frac{1}{n}\ln|m_n|\right\} \le -\tau, \quad \sigma, \tau > 0.$$

Then we have

$$\mu_2(\alpha) \le 1 + \frac{\sigma}{\tau}.$$

Now we can easily formulate our second theorem by applying lemma 5 to the following sequences:

$$F_n = T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}\left(-\frac{\mathfrak{Im}(I(x_k))}{\pi}\sqrt{2k+1}\right) =$$
$$= T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}U(x_k)\alpha_k - T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}V(x_k)\sqrt{2k+1} = X_n\alpha_k + Y_n;$$

$$G_n = T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}(2k+1)\left(2\mathfrak{Re}(I(x_k)) - 2\frac{\mathfrak{Im}(I(x_k))}{\pi}\right) =$$
$$= T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}U(x_k)\alpha_k^2 - T_{k,n}d_{(b-2a)n}\Delta_1\frac{d_{bn}}{\Delta}(2k+1)W(x_k) = X_n\alpha_k^2 + Z_n.$$

By Lemma 2, we have $X_n,\ Y_n,\ Z_n \in \mathbb{Z}$. We can also see that

$$K_2 = \lim_{n\to\infty}\frac{1}{n}\ln(T_{k,n}) = \begin{cases} -\dfrac{b-4a}{2}\ln m, & \text{if } k = 2m; \\ -\dfrac{b-4a}{2}\ln k + \dfrac{3(b-2a)}{2}\ln 2, & \text{if } k = 2m-1. \end{cases}$$

Theorem 2. *Assume that $a, b \in \mathbb{N}$ satisfies $b > 4a$, x_k is given by (5), the numbers M_1 and M_2 are defined by Statements 2 and 3, the set Ω is defined by (7), and N_2 is defined by (10).*
If $M_2 + K_2 + N_2 < 0$, then we have

$$\mu_2(\alpha_k) \leq 1 - \frac{\lim\limits_{n\to\infty} X_n}{\max\{\limsup\limits_{n\to\infty} F_n,\ \limsup\limits_{n\to\infty} G_n\}} \leq 1 - \frac{M_1 + K_2 + N_2}{M_2 + K_2 + N_2}.$$

Remark 1. In conclusion, let us give the parameter values that have been used to obtain the results presented in the beginning of this article. For each of the given values of k, the irrationality measure $\mu(\alpha_k)$ were obtained by taking $a = 1$, $b = 7$. Non-quadraticity measures $\mu_2(\alpha_k)$ have been derived by taking $a = 2$, $b = 23$ for $k = 6$ and $a = 1$, $b = 13$ for $k = 8, 10, 12$. Note that we couldn't estimate the quadratic irrationality for odd values of k because of the high growth rate of the "denominators" denoted as q_n in Lemma 5.

Remark 2. Using the same approach the author proved a few theorems. If one takes $y_k = \frac{k-1-i\sqrt{2k-1}}{k}$ in (2) instead of x_k and uses the sketch of the proof Theorem 1, one can prove

Theorem 3. *Let $\beta_k = \sqrt{2k-1}\arctan\frac{\sqrt{2k-1}}{k-1}$. Then we have*

k	$\mu(\beta_k) \leqslant$	$\mu_2(\beta_k) \leqslant$	k	$\mu(\beta_k) \leqslant$	$\mu_2(\beta_k) \leqslant$
2	4.60105...	—	8	3.66666...	14.37384...
4	3.94704...	44.87472...	10	3.60809...	12.28656...
6	3.76069...	19.19130...	12	3.56730...	11.11119...

One can read the proof of Theorem 3 in [8].

If one takes $y_k = \frac{1}{2} - i\frac{\sqrt{3}}{6}$ in (2) instead of x_k and uses the sketch of the proof Theorem 1 (one needs Lemma 4 in a somewhat different form), one can prove

Theorem 4. *For any $\varepsilon > 0$ there exists $0 < q(\varepsilon) \in \mathbb{Z}$ such that for any $q \geqslant q(\varepsilon), p_1, p_2 \in \mathbb{Z}$ we have*

$$\max\left(\left|\ln 3 - \frac{p_1}{q}\right|, \left|\frac{\pi}{\sqrt{3}} - \frac{p_2}{q}\right|\right) \geqslant q^{-3.86041\cdots-\varepsilon}.$$

One can read the proof of Theorem 4 in [8].

If one considers the integral (2), where

$$A(x) = \binom{x + 10n + 1}{10n + 1}\binom{x + 9n + 1}{8n + 1}\binom{x + 8n + 1}{6n + 1}\binom{x + 7n + 1}{4n + 1},$$

takes $x_k = \sqrt{2k+1}\ln\frac{k+1-\sqrt{2k+1}}{k}$, then using the sketch of the proof Theorem 2 one can prove

Theorem 5. *Let $\alpha_k = \sqrt{2k+1}\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}$. Then we have*

k	$\mu_2(\alpha_k) \leqslant$	k	$\mu_2(\alpha_k) \leqslant$	k	$\mu_2(\alpha_k) \leqslant$	k	$\mu_2(\alpha_k) \leqslant$	k	$\mu_2(\alpha_k) \leqslant$
2	18.5799...	8	10.3786...	13	245.5913...	16	9.03034...	19	55.9694...
4	12.8416...	10	9.86485...	14	9.23973...	17	71.3960...	20	8.7192...
6	11.2038...	12	9.50702...	15	105.4297...	18	8.86054...	21	47.1243...

One can read the proof of Theorem 5 in [9].

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